

# ADDITIVE GÖDEL LOGIC

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**ABSTRACT.** In this paper, we further develop ideas initiated in [KPT14] to study extensions of first-order Gödel Logic, called Additive Gödel Logic. A relevant model theory is developed for this logic to show that it enjoys some nice properties such as Robinson joint consistency theorem. Moreover, it is shown that the class of (ultrametric exhaustive) models with respect to elementary substructure forms an abstract elementary class.

*Keywords:* Gödel logic; Robinson joint consistency theorem; Abstract elementary class

## 1. INTRODUCTION

Extending model-theoretic techniques from classical model theory to other logics is a fashionable trend. The merit of this trend is twofold. Firstly, it can be viewed as a measurement for complexity of semantical aspects of a given logic and, secondly, can be used as an instrumental tool to verify certain fundamental logical questions. Following this, the present paper can be seen as further development initiated in [KPT14] for studying model-theoretic aspects of extensions of first-order Gödel Logic. While in [KPT14], the first-order Gödel Logic is enriched by adding countably many nullary logical constants for rational numbers, here we extend it in other way by adding a group structure on the set of truth values. This extension enables us to strengthen considerably the expressive power of the Gödel Logic. On the other hand, we will see that this strengthening does not prevent us to have nice model-theoretic properties. Therefore, this extension enjoys a balance between the expressive power, on one hand, and nice model-theoretic properties, on the other hand.

The additive Gödel logic not only involve the Gödel logic but also it includes the Łukasiewicz logic. So, this logic can be viewed as a common non-trivial extension of both of Gödel and Łukasiewicz logic. Here by non-trivial we mean that this logic is not a boolean logic [Háj98, Section 4.3]. We noted that the common extensions of known fuzzy logics are extensively studied by some authors [EGN09, EGGN07, EGN06, CEG<sup>+</sup>09, EGM11, Cin03, Cin01, EGM01, HC04, EGHN00]. Also in some papers, basic model-theoretic notions of fuzzy logics are studied [HC06, DE13, Del12, DE10, Nov03, BYBHU08].

This paper organized as follows. Next section, is devoted to introducing basic syntactical and semantical aspects of additive Gödel Logic. In third section, we show that the first-order additive Gödel Logic satisfies the compactness theorem. The crucial compactness theorem would allow us to develop some model theory for this logic and prove that the additive Gödel logic satisfies the joint Robinson consistency theorem, and furthermore it is shown that the class of (ultrametric exhaustive) models of a first-order theory  $T$  with respect to elementary substructure forms an abstract elementary class..

## 2. ADDITIVE GÖDEL LOGIC

Fuzzy logics are usually enjoy a semantic based on the unite interval of real numbers. However, the truth value set could be considered as any linear ordered structure. In this paper we want to work with a fuzzy logic, whose semantic is based on totally ordered Abelian groups.

As in classical first-order logic, we work with first-order languages. Firstly, we introduce an extension of Gödel logic, named additive Gödel logic.

**Definition 2.1.** *The first-order additive Gödel logic,  $AG\forall$ , consists of the following logical symbols:*

- (1) *Logical connectives  $\wedge, \rightarrow, \otimes, ^{-1}, \bar{1}$  and  $\perp$ .*
- (2) *Quantifiers  $\forall$  and  $\exists$ .*
- (3) *A countable set of variables  $\{x_n\}_{n \in \mathbb{N}}$ .*

*First-order languages are defined the same as classical first-order logic and are considered as non-logical symbols of  $AG\forall$ . So a language  $\tau$  is a set*

$$\tau = \{ \{ (f_i, n_{f_i}) \}_{i \in I}, \{ (P_j, n_{P_j}) \}_{j \in J} \}$$

*in which for every  $i \in I$ ,  $f_i$  is a function symbol of arity  $n_{f_i} \geq 0$  and for each  $j \in J$ ,  $P_j$  is a predicate symbol of arity  $n_{P_j} \geq 0$ . A nullary function symbol is commonly called a constant symbol.*

For a given first-order language  $\tau$ , the usual definition of  $\tau$ -terms and (atomic)  $\tau$ -formulas are considered. Free and bound variables defined as in classical first-order logic. A  $\tau$ -sentence is a  $\tau$ -formula without free variable. The set of  $\tau$ -formulas and  $\tau$ -sentences are denoted by  $Form(\tau)$  and  $Sent(\tau)$ , respectively. When there is no danger of confusion we may omit the prefix  $\tau$  and simply write a term, (atomic) formula or sentence. A theory is a set of sentences.

Further connectives are defined as follows.

$$\begin{aligned} \varphi^1 &:= \varphi \\ \varphi^n &:= \varphi^{n-1} \otimes \varphi \\ \varphi \vee \psi &:= ((\varphi \rightarrow \psi) \rightarrow \psi) \wedge ((\psi \rightarrow \varphi) \rightarrow \varphi) \\ \neg \varphi &:= \varphi \rightarrow \perp \\ \varphi \leftrightarrow \psi &:= (\varphi \rightarrow \psi) \wedge (\psi \rightarrow \varphi) \\ \top &:= \neg \perp \\ \varphi \Rightarrow \psi &:= (\psi \rightarrow \varphi) \rightarrow \psi \\ \varphi \Rrightarrow \psi &:= ((\varphi \Rightarrow \psi) \wedge \neg \neg \psi^{-1}) \vee (\psi \wedge \neg \neg \varphi^{-1}) \\ \Delta(\varphi) &:= \neg(\varphi \Rightarrow T) \\ \varphi \rightarrow_L \psi &:= \bar{1} \rightarrow (\psi \otimes \varphi^{-1}) \end{aligned}$$

The semantic of additive Gödel logic is based on totally ordered Abelian groups.

**Definition 2.2.** *Let  $(G, *, \leq)$  be a totally ordered Abelian group with an identity element  $1_G$ . Set  $\Gamma_G = \{0\} \cup G \cup \{\infty\}$ , and let*

$$\begin{aligned} \infty * 0 &= 0 * \infty = 1_G, \\ \text{for all } a \in G, \quad a * \infty &= \infty * a = \infty \text{ and } a * 0 = 0 * a = 0, \\ 0^{-1} &= \infty \text{ and } \infty^{-1} = 0. \end{aligned}$$

*Extend the order  $\leq$  on  $\Gamma_G$  such that  $0$  and  $\infty$  be the least and largest elements of  $\Gamma_G$ .*

**Remark 2.3.** *Writing  $G$  multiplicatively together with assuming a similarity relation leads to obtain a pseud ultrametric. So we choose multiplicative notion for totally ordered Abelian groups.*

For a given totally ordered Abelian group  $G$ , we consider  $\Gamma_G$  as the set of truth values, whereas 0 is the absolute falsity and  $\infty$  is the absolute truth. Define the following operators on  $\Gamma_G$ .

$$\begin{aligned} a \vee b &= \max\{a, b\} \\ a \wedge b &= \min\{a, b\} \\ a \dot{\rightarrow} b &= \begin{cases} \infty & a \leq b \\ b & a > b \end{cases} \\ d_{\min}(a, b) &= \begin{cases} \min\{a, b\} & a \neq b \\ \infty & a = b \end{cases} \end{aligned}$$

**Definition 2.4.** For a given language  $\tau$ , a  $\tau$ -structure  $\mathcal{M}$  is a nonempty set  $M$  called the universe of  $\mathcal{M}$  together with:

- a) a totally ordered Abelian group  $(G, *, \leq)$  with identity element  $1_G$  or the empty set,
- b) for any  $n$ -ary predicate symbol  $P$  of  $\tau$ , a function  $P^{\mathcal{M}} : M^n \rightarrow \Gamma_G$ , while for nullary predicate symbol,  $P^{\mathcal{M}}$  is an element of  $\Gamma_G$ ,
- c) for any  $n$ -ary function symbol  $f$  of  $\tau$ , a function  $f^{\mathcal{M}} : M^n \rightarrow M$ , while in the case of nullary function symbol,  $f^{\mathcal{M}}$  is an element of  $M$ .

We may call  $\mathcal{M}$  a  $\tau^G$ -structure. Sometimes  $\mathcal{M}$  is denoted by  $\mathcal{M} = (G, M)$ . When there is no fear of confusion, we may omit the underlying language  $\tau$  and the group symbols  $G$ , and call  $\mathcal{M}$  a structure. A structure whose underlying group is a totally ordered subgroup of  $(\mathbb{R}^{>0}, \cdot, 1)$  is called a standard structure.

For each  $\alpha \in \tau$ ,  $\alpha^{\mathcal{M}}$  is called the *interpretation* of  $\alpha$  in  $\mathcal{M}$ . The interpretation of terms defined inductively as follows.

**Definition 2.5.** For every  $n$ -tuple  $\bar{x} = x_1, x_2, \dots, x_n$  and every term  $t(\bar{x})$ , the interpretation of  $t(\bar{x})$  in  $\mathcal{M}$  is a function  $t^{\mathcal{M}} : M^n \rightarrow M$  such that

- (1) if  $t(\bar{x}) = x_i$  for  $1 \leq i \leq n$ , then  $t^{\mathcal{M}}(\bar{a}) = a_i$ ,
- (2) if  $t(\bar{x}) = f(t_1(\bar{x}), \dots, t_m(\bar{x}))$  then  $t^{\mathcal{M}}(\bar{a}) = f^{\mathcal{M}}(t_1^{\mathcal{M}}(\bar{a}), \dots, t_m^{\mathcal{M}}(\bar{a}))$ .

Similarly, the interpretation of formulas in structures is defined as follows.

**Definition 2.6.** The interpretation of a formula  $\varphi(\bar{x})$  in a  $\tau^G$ -structure  $\mathcal{M}$  is a function  $\varphi^{\mathcal{M}} : M^n \rightarrow \Gamma_G$  which is inductively determined as follows.

- (1)  $\perp^{\mathcal{M}} = 0$ ,  $\top^{\mathcal{M}} = \infty$  and  $\bar{1} = 1_G$ .
- (2) For every  $n$ -ary predicate symbol  $P$ ,

$$P^{\mathcal{M}}(t_1(\bar{a}), \dots, t_n(\bar{a})) = P^{\mathcal{M}}(t_1^{\mathcal{M}}(\bar{a}), \dots, t_n^{\mathcal{M}}(\bar{a})).$$

- (3)  $(\varphi \wedge \psi)^{\mathcal{M}}(\bar{a}) = \varphi^{\mathcal{M}}(\bar{a}) \wedge \psi^{\mathcal{M}}(\bar{a})$ .
- (4)  $(\varphi \rightarrow \psi)^{\mathcal{M}}(\bar{a}) = \varphi^{\mathcal{M}}(\bar{a}) \dot{\rightarrow} \psi^{\mathcal{M}}(\bar{a})$ .
- (5)  $(\varphi \otimes \psi)^{\mathcal{M}}(\bar{a}) = \varphi^{\mathcal{M}}(\bar{a}) * \psi^{\mathcal{M}}(\bar{a})$ .
- (6)  $(\varphi^{-1})^{\mathcal{M}}(\bar{a}) = (\varphi^{\mathcal{M}}(\bar{a}))^{-1}$ .
- (7) if  $\varphi(\bar{x}) = \forall y \psi(y, \bar{x})$  then  $\varphi^{\mathcal{M}}(\bar{a}) = \inf_{b \in M} \{\psi^{\mathcal{M}}(b, \bar{a})\}$ .
- (8) if  $\varphi(\bar{x}) = \exists y \psi(y, \bar{x})$  then  $\varphi^{\mathcal{M}}(\bar{a}) = \sup_{b \in M} \{\psi^{\mathcal{M}}(b, \bar{a})\}$ .

The suprema and infima may not exist. When all suprema and infima exist in an structure  $\mathcal{M}$ , we call  $\mathcal{M}$  a safe structure. We assume all structures to be safe hereafter. One could easily verify that:

$$\begin{aligned}
(\varphi \vee \psi)^{\mathcal{M}}(\bar{a}) &= \varphi^{\mathcal{M}}(\bar{a}) \vee \psi^{\mathcal{M}}(\bar{a}), \\
(\varphi \leftrightarrow \psi)^{\mathcal{M}}(\bar{a}) &= d_{\min}(\varphi^{\mathcal{M}}(\bar{a}), \psi^{\mathcal{M}}(\bar{a})), \\
(\neg\neg\varphi)^{\mathcal{M}}(\bar{a}) &= \begin{cases} \infty & \varphi^{\mathcal{M}}(\bar{a}) > 0, \\ 0 & \varphi^{\mathcal{M}}(\bar{a}) = 0, \end{cases} \\
(\varphi \Rightarrow \psi)^{\mathcal{M}}(\bar{a}) &= \begin{cases} \infty & \varphi^{\mathcal{M}}(\bar{a}) < \psi^{\mathcal{M}}(\bar{a}) < \infty, \\ \psi^{\mathcal{M}}(\bar{a}) & \text{otherwise,} \end{cases} \\
(\varphi \Rightarrow \psi)^{\mathcal{M}}(\bar{a}) &= \begin{cases} \infty & \varphi^{\mathcal{M}}(\bar{a}) < \psi^{\mathcal{M}}(\bar{a}), \\ 0 & \varphi^{\mathcal{M}}(\bar{a}) = \psi^{\mathcal{M}}(\bar{a}) = \infty, \\ \psi^{\mathcal{M}}(\bar{a}) & \text{otherwise,} \end{cases} \\
(\Delta(\varphi))^{\mathcal{M}}(\bar{a}) &= \begin{cases} \infty & \varphi^{\mathcal{M}}(\bar{a}) = \infty, \\ 0 & \text{otherwise,} \end{cases} \\
(\varphi \rightarrow_L \psi)^{\mathcal{M}}(\bar{a}) &= \begin{cases} \infty & \varphi^{\mathcal{M}}(\bar{a}) \leq \psi^{\mathcal{M}}(\bar{a}), \\ \psi^{\mathcal{M}}(\bar{a}) * (\varphi^{\mathcal{M}}(\bar{a}))^{-1} & \text{otherwise.} \end{cases}
\end{aligned}$$

**Remark 2.7.** The truth functionality of  $\rightarrow$ ,  $\leftrightarrow$ , and  $\wedge$  show that the logic that we work on it is an extension of Gödel logic. On the other hand, we have some additional connectives such as  $\otimes$ ,  $\rightarrow_L$ , and  $^{-1}$  whose truth functionalities acts as the truth functionality of connectives of Lukasiewicz logic (in the multiplicative notion). Note that we could define the  $\Delta$ -Bazz connective [Baa96] also.

However the expressive power of the logic is strictly stronger than Gödel logic and Lukasiewicz logic. Observe that as opposed to the Lukasiewicz logic we could express  $\varphi^{\mathcal{M}} < \psi^{\mathcal{M}}$  by  $(\varphi \Rightarrow \psi)^{\mathcal{M}} = \infty$  and also opposed to Gödel logic (and also Lukasiewicz logic) we could express  $\varphi^{\mathcal{M}} < \infty$  by  $(\varphi \Rightarrow \top)^{\mathcal{M}} = \infty$  or  $(\neg\Delta(\varphi))^{\mathcal{M}} = \infty$ .

On the other hand the expressive power is weaker than the LII logic [EGM01, Cin01, Cin03] as we could not express the product conjunction (in additive notion).

The semantical notions of satisfiability, model and entailment are defined as follows.

**Definition 2.8.** Let  $\varphi(\bar{x})$  be a  $\tau$ -formula,  $\psi$  be a  $\tau$ -sentence, and  $T$  be a  $\tau$ -theory.

- (1) If there is a  $\tau^G$ -structure  $\mathcal{M}$  and  $\bar{a} \in M^n$  such that  $\varphi^{\mathcal{M}}(\bar{a}) = \infty$ , then we call  $\varphi(\bar{x})$  a satisfiable formula. In this case, write  $\mathcal{M} \models \varphi(\bar{a})$  and call  $\mathcal{M} = (G, M)$  a model of  $\varphi(\bar{x})$ . The class of all models of  $\varphi(\bar{x})$  is denoted by  $\text{Mod}(\varphi(\bar{x}))$ .
- (2) We call  $T$  a satisfiable theory if  $\cap_{\varphi \in T} \text{Mod}(\varphi) \neq \emptyset$ . When  $\mathcal{M} \in \cap_{\varphi \in T} \text{Mod}(\varphi)$  we say that  $\mathcal{M}$  is a model of  $T$  and denote this by  $\mathcal{M} \models T$ . The class of all models of  $T$  are denoted by  $\text{Mod}(T)$ .
- (3)  $T$  is called finitely satisfiable if every finite subset of  $T$  has a model.
- (4)  $T \models \psi$ , if  $\text{Mod}(T) \subseteq \text{Mod}(\psi)$ . In this case we say that  $T$  entails  $\psi$ .
- (5) We write  $T \models^f \varphi$  if there exist a finite subset  $S$  of  $T$  such that  $S \models \varphi$ .

As in first-order logic the full theory of a  $\tau$ -structure  $\mathcal{M}$  is

$$\text{Th}_{\tau}(\mathcal{M}) = \{\varphi : \mathcal{M} \models \varphi, \varphi \in \text{Sent}(\tau)\}.$$

We may write  $\text{Th}(\mathcal{M})$  when there is no fear of confusion about the underlying language.

### 3. COMPACTNESS THEOREM

In this section, using the Henkin construction, we obtain a version of compactness theorem for additive Gödel logic.

**Definition 3.1.** Let  $T$  be a  $\tau$ -theory.

- (1)  $T$  is called a linear complete theory, if for every  $\tau$ -sentences  $\varphi$  and  $\psi$ , either  $\varphi \rightarrow \psi \in T$  or  $\psi \rightarrow \varphi \in T$ .
- (2) We say that  $T$  is Henkin, if for every  $\tau$ -formula  $\varphi(x)$  that  $T \not\models \forall x \varphi(x)$ , there exists some constant  $c$  in  $\tau$  such that  $T \not\models \varphi(c)$ .

Bellow, we prove the entailment compactness for  $AG\forall$ . Obviously, the entailment compactness implies the usual compactness theorem. The next theorem is a special case of the entailment compactness where  $T$  is linear complete and Henkin

**Theorem 3.2.** *Let  $T$  be a linear complete Henkin  $\tau$ -theory and  $\chi$  be a  $\tau$ -sentence. Then  $T \models \chi$  if and only if  $T \models^f \chi$ .*

*Proof.* From right to left direction is obvious. For the other direction, let  $T \models \chi$  and for the purpose of contradiction, suppose that  $T \not\models^f \chi$ . Define an equivalence relation  $\sim$  on the set of  $\tau$ -sentences as follows:

$$\varphi \sim \psi \text{ iff } T \models^f \varphi \leftrightarrow \psi.$$

For every  $\tau$ -sentence  $\varphi$ , let  $[\varphi]$  be the equivalence class of  $\varphi$  with respect to  $\sim$ . Let  $Lind(T)$  be the set of all equivalence classes of  $\sim$ . Define the operation  $\star$  on  $Lind(T)$  by

$$[\varphi] \star [\psi] = [\varphi \otimes \psi].$$

One can easily verify that  $G_{Lind(T)} = (Lind(T) \setminus \{[\top], [\perp]\}, \star)$  is an Abelian group with identity element  $[\bar{1}]$ . For example,  $\star$  is an associative operator, as if  $\varphi_1, \varphi_2, \varphi_3 \in Sent(\tau)$ , then

$$\begin{aligned} ([\varphi_1] \star [\varphi_2]) \star [\varphi_3] &= [\varphi_1 \otimes \varphi_2] \star [\varphi_3] = [(\varphi_1 \otimes \varphi_2) \otimes \varphi_3], \\ [\varphi_1] \star ([\varphi_2] \star [\varphi_3]) &= [\varphi_1] \star [\varphi_2 \otimes \varphi_3] = [\varphi_1 \otimes (\varphi_2 \otimes \varphi_3)]. \end{aligned}$$

Now, if  $[\varphi_1], [\varphi_2], [\varphi_3] \in G_{Lind(T)}$ , then linear completeness of  $T$  implies that

$$T \models^f \bigwedge_{i=1}^3 (\neg \neg \varphi_i \wedge \neg \Delta(\varphi_i)).$$

Hence, by an easy argument we have

$$T \models^f ((\varphi_1 \otimes \varphi_1) \otimes \varphi_1) \leftrightarrow (\varphi_1 \otimes (\varphi_1 \otimes \varphi_1)).$$

Furthermore, by defining  $<$  on  $Lind(T)$  as

$$[\varphi] < [\psi] \text{ iff } T \models^f \psi \rightarrow \varphi,$$

we make the group  $G_{Lind(T)}$  a totally ordered Abelian group such that  $Lind(T)$  is  $\Gamma_{G_{Lind(T)}}$ . The group  $G_{Lind(T)}$  is called the Lindenbaum group of  $T$ -equivalence sentences. Now, let  $CM(T)$  be the set of all closed  $\tau$ -terms, i.e., terms constructed only by constants symbols of  $\tau$ . Construct the  $\tau^{G_{Lind(T)}}$ -structure  $\mathcal{M} = (G_{Lind(T)}, CM(T))$  by setting its universe to be  $CM(T)$ , and for each  $n$ -ary function symbol  $f \in \tau$  and  $n$ -ary predicate symbol  $P \in \tau$  define

- $f^{\mathcal{M}} : CM(T)^n \rightarrow CM(T)$  by  $f^{\mathcal{M}}(t_1, \dots, t_n) = f(t_1, \dots, t_n)$ ,
- $P^{\mathcal{M}} : CM(T)^n \rightarrow Lind(T)$  by  $P^{\mathcal{M}}(t_1, \dots, t_n) = [P(t_1, \dots, t_n)]$ .

One can easily verify that for each  $\tau$ -sentence  $\varphi$ ,  $\varphi^{\mathcal{M}} = [\varphi]$ ,  $\mathcal{M} \models T$  and  $\chi^{\mathcal{M}} \neq \infty$ . □ □

$(G_{Lind(T)}, CM(T))$  is called the canonical model of the theory  $T$ . We need the following lemma to prove the entailment compactness in general case.

**Lemma 3.3.** *Let  $T$  be a  $\tau$ -theory and  $\chi$  be a  $\tau$ -sentence and  $T \not\models \chi$ . There exists a linear complete  $\tau$ -theory  $T' \supseteq T$  such that  $T' \not\models \chi$ .*

*Proof.* It is easy to see that for every  $\tau$ -sentences  $\varphi$  and  $\psi$ , either  $T \cup \{\varphi \rightarrow \psi\} \not\models \chi$  or  $T \cup \{\psi \rightarrow \varphi\} \not\models \chi$ . Now, using Zorn's lemma the desirable linear complete theory established.  $\square$   $\square$

Now, we could prove the entailment compactness in general case.

**Theorem 3.4.** *Let  $T$  be a  $\tau$ -theory and  $\chi$  be a  $\tau$ -sentence.  $T \models \chi$  if and only if  $T \models^f \chi$ .*

*Proof.* We prove the non-trivial direction. Suppose that  $T \not\models^f \chi$ . We show that there exist a language  $\tau' \supseteq \tau$  and a linear complete Henkin  $\tau'$ -theory  $T' \supseteq T$  such that  $T' \not\models \chi$ .

Let  $\chi_0 = \chi$ ,  $\tau_0 = \tau$ , and  $T_0 = T$ . On the basis of Lemma 3.3 there is a linear complete theory  $\overline{T}_0$  containing  $T_0$  such that  $\overline{T}_0 \not\models \chi_0$ . We extend the language  $\tau_0$  by adding a new nullary predicate symbol  $\chi_1$  and new constant symbols  $\{c_\varphi : \varphi(x) \in \text{Form}(\tau_0)\}$  and let  $\tau_1 = \tau \cup \{\chi_1\} \cup \{c_\varphi : \overline{T}_0 \not\models \forall x \varphi(x)\}$ . Subsequently put

$$T_1 = \overline{T}_0 \cup \{\chi_0 \rightarrow \chi_1\} \cup \{\varphi(c_\varphi) \rightarrow \chi_1 : \overline{T}_0 \not\models \forall x \varphi(x)\}.$$

Now, we show that  $T_1 \not\models \chi_1$ . To this end, let  $U$  be a finite subset of  $\overline{T}_0$  and  $S = U \cup \{\chi_0 \rightarrow \chi_1\} \cup \{\varphi_i(c_{\varphi_i}) \rightarrow \chi_1 : \overline{T}_0 \not\models \forall x \varphi_i(x)\}_{i=1}^m$ . Since  $\overline{T}_0$  is linear complete and  $\overline{T}_0 \not\models \chi_0$  and also  $\overline{T}_0 \not\models \forall x \varphi_i(x)$  for  $1 \leq i \leq m$  it follows that  $\overline{T}_0 \not\models \chi_0 \vee (\bigvee_{i=1}^m \forall x \varphi_i(x))$ . Hence, there is a  $\tau_0$ -structure  $\mathcal{M} \models U$  such that  $\max\{\chi_0^{\mathcal{M}}, (\forall x \varphi_1(x))^{\mathcal{M}}, \dots, (\forall x \varphi_m(x))^{\mathcal{M}}\} = g < \infty$ . Now, interpreting  $\chi_1^{\mathcal{M}}$  by  $g$ , making  $\mathcal{M}$  as a  $\tau_1$ -structure such that  $\mathcal{M} \models S$  and  $\chi_1^{\mathcal{M}} = g < \infty$ .

By iterating the above construction, we get sequence  $\tau_0 \subseteq \tau_1 \subseteq \dots$  of first-order languages,  $T_0 \subseteq T_1 \subseteq \dots \subseteq T_n \subseteq \dots$  of  $\tau_n$ -theories, and  $\{\chi_n\}_{n=0}^\infty$  of  $\tau_n$ -sentences such that for each  $n \geq 0$ ,  $T_n \not\models \chi_n$  and  $T_{n+1} \models \chi_n \rightarrow \chi_{n+1}$ . Set  $\tau' = \bigcup_{n \geq 0} \tau_n$  and let  $T_\infty = \bigcup_{n \geq 0} T_n$ . Clearly,  $T_\infty \not\models \chi$ . Thus, on the basis of Lemma 3.3 there exists a linear complete theory  $T'$  containing  $T_\infty$  such that  $T' \not\models \chi$ . Obviously,  $T'$  is a Henkin  $\tau'$ -theory. It follows from Theorem 3.2 that  $T' \not\models \chi$ . So,  $T \not\models^f \chi$ .  $\square$   $\square$

The compactness theorem immediately follows from the above theorem.

**Corollary 3.5.** *(Compactness Theorem) A theory  $T$  is satisfiable if and only if it is finitely satisfiable.*

**Remark 3.6.** *Note that if  $T$  is finitely satisfiable by  $G$ -models, then it is not necessarily satisfiable by a  $G$ -model, while by the above corollary  $T$  is satisfiable by a  $G'$ -model for some totally ordered Abelian group  $G'$ . To see this, let  $\mathcal{L} = \{\epsilon, \rho\}$  be a relational language consisting of two nullary predicate symbols. Set,*

$$T = \{\bar{1} \Rightarrow \rho, \epsilon \Rightarrow \top\} \cup \{\rho^n \Rightarrow \epsilon\}_{n \in \mathbb{N}}.$$

*$T$  is finitely satisfiable by standard models, but it has no standard model. On the other hand, observe that by compactness theorem  $T$  is satisfiable. For example, if we take  $G = (\mathbb{R}^{>0})^2$  with the lexicographical ordering and the componentwise multiplication, then  $T$  has a  $G$ -model.*

One could naturally ask whether any satisfiable theory has a standard model.

**Conjecture 3.7.** *If  $T$  is a finite satisfiable theory, then it has a standard model.*

## 4. SOME MODEL THEORY

In this section, some basic model theoretic concepts of  $AG\forall$  is studied. Various model theoretic definitions such as elementary equivalence, elementary embedding, substructure, and diagram are studied recently in the context of mathematical fuzzy logics [CH10, HC06, DE10, Del12, DE13].

In this paper, we assume that the underlying language  $\tau$  contains a binary predicate symbol which reflects the properties of the equality relation. This assumption is necessary, since most model theoretic results can not be achieved without the equality relation.

4.1.  $AG\forall$  with the Equality Relation.

In the rest of this section, fix a first-order language  $\tau_e$  including a binary predicate symbol  $e$ . This predicate plays the same role as the equality relation in classical first-order logic. The essential properties of the equality relation are the similarity axioms, i.e.,

$$\begin{aligned} & \forall x (x = x), \\ & \forall x \forall y (x = y \rightarrow y = x), \\ & \forall x \forall y \forall z ((x = y \wedge y = z) \rightarrow x = z). \end{aligned}$$

Let  $\mathcal{M}$  be a  $\tau_e$ -structure which models the following similarity axioms.

$$\{\forall x e(x, x), \forall x \forall y (e(x, y) \rightarrow e(y, x)), \forall x \forall y \forall z ((e(x, y) \wedge e(y, z)) \rightarrow e(x, z))\}.$$

Then, for all  $a, b, c \in \mathcal{M}$ ,

$$\begin{aligned} e^{\mathcal{M}}(a, a) &= \infty, \\ e^{\mathcal{M}}(a, b) &= e^{\mathcal{M}}(b, a), \\ e^{\mathcal{M}}(a, b) &\geq \min\{e^{\mathcal{M}}(a, c), e^{\mathcal{M}}(b, c)\}. \end{aligned}$$

So, the interpretation of  $e^{-1}$  in  $\mathcal{M}$  is as like as a pseudo-ultrametric on the universe of  $\mathcal{M}$  (a pseudo-ultrametric in which for all  $a, b, c \in M$ ,  $(e^{-1})^{\mathcal{M}}(a, b) \leq \max\{(e^{-1})^{\mathcal{M}}(a, c), (e^{-1})^{\mathcal{M}}(b, c)\}$ ).

**Definition 4.1.** Let  $\mathcal{M} = (G, M)$  be a  $\tau_e$ -structure. We call  $\mathcal{M}$  An ultrametric structure, whenever for all  $a, b, c \in M$

- $(e^{-1})^{\mathcal{M}}(a, b) = 0$  if and only if  $a = b$ ,
- $(e^{-1})^{\mathcal{M}}(a, b) = (e^{-1})^{\mathcal{M}}(b, a)$ ,
- $(e^{-1})^{\mathcal{M}}(a, b) \leq \max\{(e^{-1})^{\mathcal{M}}(a, c), (e^{-1})^{\mathcal{M}}(b, c)\}$ .

To simplify the notions, for a  $\tau_e$ -ultrametric structure  $\mathcal{M}$  we denote  $e^{-1}$  by  $d$ .

**Example 4.2.** Any first-order structure could be viewed as an ultrametric structure. As we expect, any ultrametric space  $(M, d)$  is an ultrametric structure. Also Any normed field (valued field) is an ultrametric structure.

**Definition 4.3.** Let  $T$  be a  $\tau_e$ -theory and  $\varphi$  be a  $\tau_e$ -sentence.

- (1)  $T$  is called an  $m$ -satisfiable theory if there is an ultrametric structure  $\mathcal{M} \models T$ .
- (2) We call  $T$  finitely  $m$ -satisfiable whenever every finite subset of  $T$  has an ultrametric model.
- (3)  $T \models_m \varphi$  if each ultrametric model of  $T$ , models  $\varphi$ .
- (4)  $T \not\models_m^f \varphi$  if there is a finite subset  $S$  of  $T$  such that  $S \models_m \varphi$ . Otherwise we write  $T \models_m^f \varphi$ .

The ultrametric version of entailment compactness could be established as follows.

**Theorem 4.4.** Let  $T$  be a  $\tau_e$ -theory and  $\chi$  be a  $\tau_e$ -sentence.  $T \models_m \chi$  if and only if  $T \models_m^f \chi$ .

*Proof.* The proof is similar to the proof of Theorem 3.4. By the same way as the proof of Theorem 3.4 we could assume that  $T$  is a linear complete Henkin  $\tau_e$ -theory. Let  $G_{Lind(T)}$  be the Lindenbaum group of  $T$ -equivalence sentences introduced in Theorem 3.2. Note that here  $Lind(T)$  is the set of all equivalence classes of the relation  $\sim$  on  $Sent(\tau)$  which is defined by

$$\varphi \sim \psi \text{ if and only if } T \models_m^f \varphi \leftrightarrow \psi.$$

However, the definition of the order on  $Lind(T)$  does not change. So, for  $\varphi, \psi \in Sent(\tau)$ ,  $[\varphi] \leq [\psi]$  if and only if  $T \models_m^f \psi \rightarrow \varphi$ .

Define an equivalence relation on the set of all closed  $\tau_e$ -terms as follows.

$$t_1 \sim t_2 \text{ if and only if } T \models_m^f e(t_1, t_2)$$

Let  $\langle t \rangle$  be equivalence class of  $t$  and suppose that  $CM_m(T)$  be the set of equivalence classes of  $\sim$ . The canonical ultrametric structure  $(G_{Lind(T)}, CM_m(T))$  of  $T$  is constructed as follows:

- For each  $n$ -ary function symbol  $f$  define the function  $f^{\mathcal{M}} : M^n \rightarrow M$  by  $f^{\mathcal{M}}(\langle t_1 \rangle, \dots, \langle t_n \rangle) = \langle f(t_1, \dots, t_n) \rangle$ .
- For each  $n$ -ary predicate symbol  $P$  define  $P^{\mathcal{M}} : M^n \rightarrow \Gamma_{G_{Lind(T)}}$  by  $P^{\mathcal{M}}(\langle t_1 \rangle, \dots, \langle t_n \rangle) = [P(t_1, \dots, t_n)]$ .

Note that  $f^{\mathcal{M}}$  is well-defined. Indeed, if for  $1 \leq i \leq n_f$ ,  $\langle t_i \rangle = \langle t'_i \rangle$ , then  $T \models_m^f e(t_i, t'_i)$  and by linear completeness of  $T$  we have  $T \models_m^f \bigwedge_{i=1}^{n_f} e(t_i, t'_i)$ . Hence, there is a finite subset  $S$  of  $T$  such that for each ultrametric model  $\mathcal{N} \models S$ ,  $(t_i)^{\mathcal{N}} = (t'_i)^{\mathcal{N}}$  for  $1 \leq i \leq n_f$ . So,

$$f^{\mathcal{N}}((t_1)^{\mathcal{N}}, \dots, (t_{n_f})^{\mathcal{N}}) = f^{\mathcal{N}}((t'_1)^{\mathcal{N}}, \dots, (t'_{n_f})^{\mathcal{N}})$$

i.e.,

$$\mathcal{N} \models e(f(t_1, \dots, t_n), f(t'_1, \dots, t'_n)).$$

But, then as  $\mathcal{N}$  is any arbitrary ultrametric model of  $S$  we have

$$T \models_m^f e(f(t_1, \dots, t_n), f(t'_1, \dots, t'_n)).$$

A similar argument show that  $P^{\mathcal{M}}$  is well-defined and this complete the proof.  $\square$   $\square$

**Corollary 4.5.** *A theory  $T$  is finitely  $m$ -satisfiable if and only if it is  $m$ -satisfiable.*

#### 4.2. Basic Notions of Model Theory.

The definition of elementary equivalent models in classical first-order logic is based on satisfactory of the same sentences by models. In the case of many-valued logic the same definition could be chosen.

**Definition 4.6.** *Let  $\mathcal{M} = (G, M)$  and  $\mathcal{N} = (H, N)$  be two  $\tau$ -structures.*

- (1)  *$\mathcal{M}$  and  $\mathcal{N}$  are elementary equivalent,  $\mathcal{M} \equiv \mathcal{N}$ , if  $Th(\mathcal{M}) = Th(\mathcal{N})$ .*
- (2) *If  $A \subseteq M, N$  and  $\tau(A)$  be the expansion of  $\tau$  by adding some new constant symbols  $c_a$  for each  $a \in A$ , then  $\mathcal{M}$  and  $\mathcal{N}$  can be viewed naturally as  $\tau(A)$ -structures. We say that  $\mathcal{M}$  and  $\mathcal{N}$  are elementary equivalent over  $A$ ,  $\mathcal{M} \equiv_A \mathcal{N}$ , if  $Th_{\tau(A)}(\mathcal{M}) = Th_{\tau(A)}(\mathcal{N})$ .*

A structure whose underlying group does not contain any unnecessary element is called an exhaustive structure. This notion firstly appeared in [HC05].

**Definition 4.7.** *For a  $\tau$ -structure  $\mathcal{M} = (G, M)$  let  $Gr(\mathcal{M})$  or  $Gr((G, M))$  be the ordered subgroup of truth values of all  $\tau$ -formulas, i.e.,*

$$Gr(\mathcal{M}) = \{\varphi^{\mathcal{M}}(\bar{a}) : \varphi \in Form(\tau), \bar{a} \subseteq M\} \setminus \{0, \infty\}.$$

$\mathcal{M} = (G, M)$  is called an exhaustive structure if  $G = Gr(\mathcal{M})$ .

The definition of elementary embedding is based on the equality of truth values of formulas [HC06, BYBH08]. for example if  $\mathcal{M}$  and  $\mathcal{N}$  are two  $\tau$ -structures with the same set of truth values, then  $\mathcal{M}$  is elementary embedded in  $\mathcal{N}$  if there is an injection  $h : M \rightarrow N$  such that



$$\varphi^{\mathcal{M}}(a_1, a_2, \dots, a_n) = \varphi^{\mathcal{N}}(h(a_1), h(a_2), \dots, h(a_n))$$

For additive Gödel logic, we give more suitable definition.

**Definition 4.8.** Let  $\mathcal{M} = (G, M)$  and  $\mathcal{N} = (H, N)$  be  $\tau$ -structures. We say that  $\mathcal{M}$  is elementary embedded in  $\mathcal{N}$ , if there are an injection  $h : M \rightarrow N$  and a strict order preserving group homeomorphism  $T : G \rightarrow H$  such that:

- $h(f^{\mathcal{M}}(a_1, \dots, a_{n_f})) = f^{\mathcal{N}}(h(a_1), \dots, h(a_{n_f}))$ , for all function symbols  $f \in \tau$  and  $\bar{a} \in M^{n_f}$ ,
- $T(\varphi^{\mathcal{M}}(a_1, a_2, \dots, a_n)) = \varphi^{\mathcal{N}}(h(a_1), h(a_2), \dots, h(a_n))$ , for all  $\varphi \in \text{Form}(\tau)$  and  $\bar{a} \subseteq M$ .

We call  $(h, T) : \mathcal{M} \hookrightarrow_{\tau} \mathcal{N}$  an elementary embedding from  $\mathcal{M}$  into  $\mathcal{N}$ .  $\mathcal{M}$  and  $\mathcal{N}$  are called isomorphic,  $\mathcal{M} \cong \mathcal{N}$ , if  $T$  is a group isomorphism and there are two elementary embeddings  $(h, T) : \mathcal{M} \hookrightarrow_{\tau} \mathcal{N}$  and  $(j, T^{-1}) : \mathcal{N} \hookrightarrow_{\tau} \mathcal{M}$ . Obviously, in this case  $h$  is a one-to-one correspondence and we call  $(h, T)$  an isomorphism.

Clearly, the isomorphism relation between  $\tau$ -structures is an equivalence relation.

**Lemma 4.9.** Let  $\mathcal{M}$  and  $\mathcal{N}$  be exhaustive structures and there is an injection  $h : M \rightarrow N$  such that

- $h(f^{\mathcal{M}}(a_1, \dots, a_{n_f})) = f^{\mathcal{N}}(h(a_1), \dots, h(a_{n_f}))$ , for all function symbols  $f \in \tau$  and  $\bar{a} \in M^{n_f}$ ,
- $\mathcal{M} \models \varphi(a_1, a_2, \dots, a_n)$  if and only if  $\mathcal{N} \models \varphi(h(a_1), h(a_2), \dots, h(a_n))$ , for all  $\varphi \in \text{Form}(\tau)$  and  $\bar{a} \subseteq M$ .

There is a strict order preserving group homeomorphism  $I_{\mathcal{MN}} : \text{Gr}(\mathcal{M}) \rightarrow \text{Gr}(\mathcal{N})$  such that  $(h, I_{\mathcal{MN}})$  is an elementary embedding from  $\mathcal{M}$  into  $\mathcal{N}$ .

*Proof.* Obviously,  $I_{\mathcal{MN}}(\varphi^{\mathcal{M}}(a_1, \dots, a_n)) = \varphi^{\mathcal{N}}(h(a_1), \dots, h(a_n))$  does the job.  $\square$   $\square$

**Remark 4.10.** If  $\mathcal{M}$  and  $\mathcal{N}$  are exhaustive ultrametric  $\tau_e$ -structures and there exist a function  $h : M \rightarrow N$  such that

$$\mathcal{M} \models \varphi(a_1, a_2, \dots, a_n) \text{ if and only if } \mathcal{N} \models \varphi(h(a_1), h(a_2), \dots, h(a_n)), \text{ for all } \varphi \in \text{Form}(\tau) \text{ and } \bar{a} \subseteq M,$$

then one could easily see that  $(h, I_{\mathcal{MN}})$  is an elementary embedding.

One of the nice properties of model theory of first-order logic is "amalgamating many structures into one structure". To study this property in additive Gödel logic, as in classical first-order logic, we need the method of diagram.

**Definition 4.11.** Let  $\mathcal{M} = (G, M)$  be a  $\tau$ -structure. The elementary diagram of  $\mathcal{M}$  is

$$\text{eddiag}_{\tau}(\mathcal{M}) = \text{Th}_{\tau(M)}(\mathcal{M}).$$

We may write  $\text{eddiag}(\mathcal{M})$  when there is no danger of confusion about the underlying language.

An important property of elementary diagram in classical first-order logic is describing the structure, i.e., if  $\mathcal{M}$  be a  $\tau$ -structure and  $\mathcal{N}$  be a  $\tau(M)$ -structure such that  $\mathcal{N} \models \text{eddiag}(\mathcal{M})$ , then there is an elementary embedding  $j : \mathcal{M} \hookrightarrow_{\tau} \mathcal{N}$ .

Below, we show that the elementary diagram of an exhaustive ultrametric structure, fully describe the structure.

**Lemma 4.12.** Let  $\mathcal{M} = (G, M)$  be an exhaustive ultrametric  $\tau_e$ -structure. Suppose for some exhaustive ultrametric  $\tau_e(M)$ -structure  $\mathcal{N} = (H, N)$ ,  $\mathcal{N} \models \text{eddiag}(\mathcal{M})$ . Then, there is a  $\tau_e$  elementary embedding from  $\mathcal{M}$  into  $\mathcal{N}$ .

*Proof.* Define  $j : M \rightarrow N$  by  $j(m) = m^{\mathcal{N}}$ . Obviously,  $j$  is an injection. Indeed, if  $a$  and  $b$  are two distinct element of  $M$ , then  $(\perp \Rightarrow d(a, b)) \in \text{eddiag}(\mathcal{M})$ . Thus,  $\mathcal{N} \models \perp \Rightarrow d(a, b)$ . So,  $d^{\mathcal{N}}(a^{\mathcal{N}}, b^{\mathcal{N}}) > 0$ , i.e.,  $j(a) \neq j(b)$ .

On the other hand, if for some  $n$ -ary function symbol  $f$  and element  $b \in M$ ,  $f^{\mathcal{M}}(a_1, \dots, a_n) = b$ , then  $d(f(a_1, \dots, a_n), b) \in \text{eddiag}(\mathcal{M})$ . So,  $\mathcal{N} \models d(f(a_1, \dots, a_n), b)$ , that is

$$f^{\mathcal{N}}(j(a_1), \dots, j(a_n)) = f^{\mathcal{N}}(a_1^{\mathcal{N}}, \dots, a_n^{\mathcal{N}}) = b^{\mathcal{N}} = j(b) = j(f^{\mathcal{M}}(a_1, \dots, a_n)).$$

Furthermore, if  $\mathcal{M} \models \varphi(a_1, \dots, a_n)$  for a  $\tau_e$ -formula  $\varphi(x_1, \dots, x_n)$  and  $\bar{a} \in M^n$ , then  $\varphi(a_1, \dots, a_n) \in \text{eddiag}(\mathcal{M})$ . So,  $\mathcal{N} \models \varphi(j(a_1), \dots, j(a_n))$ . Conversely, if  $\mathcal{N} \models \varphi(j(a_1), \dots, j(a_n))$  for a  $\tau_e$ -formula  $\varphi(x_1, \dots, x_n)$  and  $\bar{a} \in M^n$ , then  $\mathcal{M} \models \varphi(a_1, \dots, a_n)$ , since otherwise  $\neg \Delta(\varphi(a_1, \dots, a_n)) \in \text{eddiag}(\mathcal{M})$ . But, this contradicts with  $\mathcal{N} \models \varphi(j(a_1), \dots, j(a_n))$ . Now, by Lemma 4.9  $(j, I_{\mathcal{M}\mathcal{N}})$  is the desirable elementary embedding.  $\square$

Now, we prove elementary amalgamation over ultrametric structures.

**Theorem 4.13.** *Let  $\mathcal{A} = (G_A, A)$ ,  $\mathcal{B} = (G_B, B)$  and  $\mathcal{M} = (G_M, M)$  be three exhaustive ultrametric  $\tau_e$ -structures. Suppose also,  $(j, I_{\mathcal{M}\mathcal{A}}) : \mathcal{M} \hookrightarrow_{\tau_e} \mathcal{A}$  and  $(k, I_{\mathcal{M}\mathcal{B}}) : \mathcal{M} \hookrightarrow_{\tau_e} \mathcal{B}$  are elementary embeddings. Then, there are exhaustive ultrametric  $\tau_e$ -structure  $\mathcal{N} = (G_N, N)$  and elementary embeddings  $(j_1, I_{\mathcal{A}\mathcal{N}}) : \mathcal{A} \hookrightarrow_{\tau_e} \mathcal{N}$  and  $(k_1, I_{\mathcal{B}\mathcal{N}}) : \mathcal{B} \hookrightarrow_{\tau_e} \mathcal{N}$  such that  $j_1 \circ j = k_1 \circ k$ .*

*Proof.* Let  $\tau_A = \tau_e(M) \cup \{c_a : a \in A \setminus j(M)\}$ ,  $\tau_B = \tau_e(M) \cup \{c_b : b \in B \setminus j(M)\}$  and  $\tau' = \tau_A \cup \tau_B$ . Without loss of generality, we may assume that  $\tau_A \cap \tau_B = \tau_e(M)$ . One can naturally interpret the new constants  $c_a$  and  $c_m$ , for  $a \in A$  and  $m \in M$  inside the ultrametric  $\tau_A$ -structure  $\mathcal{A}$  by  $a$  and  $j(m)$ , respectively. Similarly, for  $b \in B$  and  $m \in M$  interpret  $c_b$  and  $c_m$  inside  $\mathcal{B}$  by  $b$  and  $k(m)$ , respectively. We want to show that  $\text{eddiag}(\mathcal{A}) \cup \text{eddiag}(\mathcal{B})$  is an  $m$ -satisfiable  $\tau'$ -theory.

For a given  $\varphi(c_{a_1}, \dots, c_{a_i}, c_{m_1}, \dots, c_{m_j}) \in \text{eddiag}(\mathcal{A})$ , we have

$$\varphi^{\mathcal{A}}(c_{a_1}, \dots, c_{a_i}, c_{m_1}, \dots, c_{m_j}) = \infty, \text{ and } (\varphi^{-1})^{\mathcal{A}}(c_{a_1}, \dots, c_{a_i}, c_{m_1}, \dots, c_{m_j}) = 0.$$

Thus,

$$\mathcal{A} \models \exists \bar{x} (\varphi(\bar{x}, c_{m_1}, \dots, c_{m_j}) \wedge (\varphi^{-1}(\bar{x}, c_{m_1}, \dots, c_{m_j}) \rightarrow \perp)).$$

Now, since  $j : \mathcal{M} \hookrightarrow \mathcal{A}$  and  $k : \mathcal{M} \hookrightarrow \mathcal{B}$  are elementary embeddings, we have

$$\mathcal{B} \models \exists \bar{x} (\varphi(\bar{x}, c_{m_1}, \dots, c_{m_j}) \wedge (\varphi^{-1}(\bar{x}, c_{m_1}, \dots, c_{m_j}) \rightarrow \perp)).$$

Hence,  $\sup_{\bar{b} \in B^i} (\varphi^{\mathcal{B}}(\bar{b}, c_{m_1}, \dots, c_{m_j}) \wedge ((\varphi^{-1})^{\mathcal{B}}(\bar{b}, c_{m_1}, \dots, c_{m_j}) \rightarrow 0)) = \infty$ , i.e., for any  $g \in G_B$  there exists an  $i$ -tuple  $\bar{b} \in B^i$  such that

$$\varphi^{\mathcal{B}}(\bar{b}, c_{m_1}, \dots, c_{m_j}) \geq g \text{ and } ((\varphi^{-1})^{\mathcal{B}}(\bar{b}, c_{m_1}, \dots, c_{m_j}) \rightarrow 0) \geq g.$$

So, for some  $i$ -tuple  $\bar{b} \in B^i$ ,  $(\varphi^{-1})^{\mathcal{B}}(\bar{b}, c_{m_1}, \dots, c_{m_j}) = 0$ . Whence, by definition of  $\infty$ ,

$$\varphi^{\mathcal{B}}(\bar{b}, c_{m_1}, \dots, c_{m_j}) = \infty.$$

Thus,

$$\mathcal{B} \models \text{eddiag}(\mathcal{B}) \cup \{\varphi(c_{a_1}, \dots, c_{a_i}, c_{m_1}, \dots, c_{m_j})\}$$

where  $\varphi(c_{a_1}, \dots, c_{a_i}, c_{m_1}, \dots, c_{m_j})$  is an arbitrary element of  $\text{eddiag}(\mathcal{A})$ . A similar argument shows that  $\text{eddiag}(\mathcal{B}) \cup \text{eddiag}(\mathcal{A})$  is finitely  $m$ -satisfiable. So, by compactness theorem  $\text{eddiag}(\mathcal{B}) \cup \text{eddiag}(\mathcal{A})$  is  $m$ -satisfiable. Now, any exhaustive ultrametric model  $\mathcal{N} = (G_N, N) \models \text{eddiag}(\mathcal{A}) \cup \text{eddiag}(\mathcal{B})$  fulfills the requirement.  $\square$

**Definition 4.14.** *A  $\tau^G$ -structure  $\mathcal{M}$  is called an elementary substructure of a  $\tau^H$ -structure  $\mathcal{N}$  (or  $\mathcal{N}$  is an elementary extension of  $\mathcal{M}$ ) if  $M \subseteq N$  and the inclusion map from  $M$  into  $N$  together with  $I_{\mathcal{M}\mathcal{N}}$  be an elementary embedding. We denote this by  $\mathcal{M} \prec \mathcal{N}$ .*

Bellow, we see that the class of exhaustive structures is closed under the union of elementary chains.

**Theorem 4.15.** *Let  $\{\mathcal{M}_i\}_{i=0}^\infty$  be a sequence of exhaustive  $\tau$ -structures such that  $\mathcal{M}_i \prec \mathcal{M}_{i+1}$  for each  $i \geq 1$ . There exists a unique  $\tau$ -structure  $\mathcal{M}$  with the underlying universe  $M = \bigcup_{i=1}^\infty M_i$  such that  $\mathcal{M}_i \prec \mathcal{M}$  for each  $i \geq 1$ .*

*Proof.* Let  $\tau_1 = \tau \cup \{c_m : m \in M_1\}$ . For each  $i \geq 2$  set  $\tau_i = \tau_{i-1} \cup \{c_m : m \in M_i \setminus M_{i-1}\}$  and put  $\tau_\infty = \bigcup_{i=1}^\infty \tau_i$ . Consider  $\mathcal{M}_i$  as a  $\tau_i$ -structure by interpreting each  $c_m \in \tau_i$  by  $m$ , itself. Fix  $E_i = Th_{\tau_i}(\mathcal{M}_i)$ .

Obviously,  $\Sigma = \bigcup_{i=1}^\infty E_i$  is finitely satisfiable. Thus, by the compactness theorem,  $\Sigma$  is satisfiable. We show that there is an exhaustive model  $\mathcal{M}$  of  $\Sigma$  such that its underlying universe is  $\bigcup_{i=1}^\infty M_i$  and for each  $i \geq 1$ ,  $\mathcal{M}_i \prec \mathcal{M}$ . To this end, we prove the followings:

- (1)  $\Sigma$  is a linear complete Henkin theory.
- (2) The underlying universe of the canonical model  $(G_{Lind(\Sigma)}, CM(\Sigma))$  of  $\Sigma$  is identical to  $\bigcup_{i=1}^\infty M_i$ .

The linear completeness is obvious. Now, for every  $\tau_\infty$ -formula  $\varphi(x, \bar{c})$ , assume that  $n_\varphi$  be the least natural number such that  $\varphi(x, \bar{c}) \in Form(\tau_{n_\varphi})$ . If  $\Sigma \not\models \forall x \varphi(x, \bar{c})$  then  $\forall x \varphi(x, \bar{c}) \notin \Sigma$  and consequently  $\forall x \varphi(x, \bar{c}) \notin E_{n_\varphi}$ . Thus  $(\forall x \varphi(x, \bar{c}))^{\mathcal{M}_{n_\varphi}} < \infty$  which implies that there exists an element  $b \in M_{n_\varphi}$  such that  $\varphi^{\mathcal{M}_{n_\varphi}}(b, \bar{c}^{\mathcal{M}_{n_\varphi}}) < \infty$ . This implies,  $\neg \Delta(\varphi(c_b, \bar{c})) \in E_{n_\varphi} \subseteq \Sigma$  and therefore  $\Sigma \not\models \varphi(c_b, \bar{c})$ . It follows that  $\Sigma$  is Henkin.

On the other hand, in the light of Theorem 3.2 the underlying universe of the canonical model of  $\Sigma$  is the set of closed  $\tau_\infty$ -terms, which can be easily seen that it is identical to  $\bigcup_{i=1}^\infty M_i$ .

Now, as for each  $j \geq 1$ ,  $\mathcal{M}_j$  is exhaustive, the function  $T : Gr(\mathcal{M}_j) \rightarrow G_{Lind(\Sigma)}$  defined by  $T(\varphi^{\mathcal{M}_j}(m_1, \dots, m_n)) = [\varphi(c_{m_1}, \dots, c_{m_n})]$  is a well-defined strict order preserving group homeomorphism. Thus, if  $i$  is the inclusion map from  $M_j$  into  $\bigcup_{i=1}^\infty M_i$ , then  $(i, T)$  is an elementary embedding from  $\mathcal{M}_j$  into  $\mathcal{M}$ , that is  $\mathcal{M}_j \prec \mathcal{M}$ .

Finally, if  $\mathcal{P}$  is another exhaustive  $\tau$ -structure with the same underlying universe  $\bigcup_{i=1}^\infty M_i$  such that for each  $i \geq 1$ ,  $\mathcal{M}_i \prec \mathcal{P}$ , then a straightforward argument as above paragraph show that  $\mathcal{P} \cong \mathcal{M}$ .  $\square \quad \square$

The model  $(G_{Lind(\Sigma)}, \bigcup_{i=1}^\infty M_i)$  is denoted by  $\bigcup_{i=1}^\infty \mathcal{M}_i$ .

**Lemma 4.16.** *Let  $\mathcal{M}_1 \prec \mathcal{M}_2 \prec \dots$  be a sequence of exhaustive  $\tau$ -structures,  $\mathcal{N} = (N, G_{\mathcal{N}})$  be an exhaustive  $\tau$ -structure, and  $\mathcal{M}_i \prec \mathcal{N}$ , for all  $i \geq 1$ . Then,  $\bigcup_{i=1}^\infty \mathcal{M}_i \prec \mathcal{N}$ .*

*Proof.* Assume that  $i$  is the inclusion map. For each  $j \geq 1$ , let  $(i, T_j)$  be an elementary embedding from  $\mathcal{M}_j$  into  $\mathcal{N}$ . Define  $T : G_{Lind(\Sigma)} \rightarrow G_{\mathcal{N}}$  by  $T([\varphi(c_{m_1}, \dots, c_{m_n})]) = T_{n_\varphi}(m_1, \dots, m_n)$  where  $n_\varphi$  is introduced in Theorem 4.15. One could easily verify that  $T$  is a well-defined strict order preserving group homeomorphism. Now  $(i, T)$  is an elementary embedding from  $\bigcup_{i=1}^\infty \mathcal{M}_i$  into  $\mathcal{N}$ .  $\square \quad \square$

To prove the downward Löwenheim-Skolem theorem we use the following definition and lemma.

**Definition 4.17.** *Let  $\tau$  be a first-order language and*

$$\mathcal{M} = (G, M, \{R^{\mathcal{M}}\}, \{f^{\mathcal{M}}\}, \{c^{\mathcal{M}}\})$$

*be a  $\tau$ -structure. Set  $\tau_G = \tau \cup \{\leq, *, ^{-1}\} \cup \{0, 1, \infty\}$  where each  $n$ -ary predicate symbol of  $\tau$  assume to be an  $n+1$ -ary predicate symbol of  $\tau_G$ ,  $\leq$  is a binary predicate symbol,  $*$  is a binary function symbol,  $^{-1}$  is a unary function symbol, and  $\{0, 1, \infty\}$  are new constant symbols. Construct a classic first-order two-sorted  $\tau_G$ -structure as follows.*

$$\mathcal{M}_G = (\langle \Gamma_G, M \rangle, \leq_{\mathcal{M}_G}, *_{\mathcal{M}_G}, ^{-1}_{\mathcal{M}_G}, 0^{\mathcal{M}_G}, 1^{\mathcal{M}_G}, \infty^{\mathcal{M}_G}, \{S^{\mathcal{M}_G} : S \in \tau\}),$$

*where*

- $0^{\mathcal{M}_G} = 0, 1^{\mathcal{M}_G} = 1_G, \infty^{\mathcal{M}_G} = \infty,$
- $\leq_{\mathcal{M}_G} = \{(g, h) \in \Gamma_G \times \Gamma_G : g \leq h\},$
- $g *_{\mathcal{M}_G} h = g * h,$

- $g^{-1}{}^{\mathcal{M}_G} = g^{-1}$ ,
- $c^{\mathcal{M}_G} = c^{\mathcal{M}}$ ,
- $f^{\mathcal{M}_G} = f^{\mathcal{M}}$ ,
- $R^{\mathcal{M}_G} = \{(\bar{a}, g) : \bar{a} \in M^n, g \in \Gamma_G, R^{\mathcal{M}}(\bar{a}) = g\}$ .

**Lemma 4.18.** *With the notions of Definition 4.17, for every  $\tau$ -sentence  $\varphi$ , there is a  $\tau_G$ -formula  $\varphi_G(g)$  such that*

$$\mathcal{M} \models \varphi \text{ if and only if } \mathcal{M}_G \models \exists g (\varphi_G(g) \wedge (g = \infty)).$$

*Proof.* By induction on the complexity of  $\tau$ -formulas for a given  $\varphi(\bar{x})$ , We introduce  $\varphi_G(\bar{x}, g)$ .

- For  $\perp$  let  $\varphi_G(g) := (g = 0)$ . Similarly, for  $\top$  let  $\varphi_G(g) := (g = \infty)$  and for  $\bar{1}$  let  $\varphi_G(g) := (g = 1)$ .
- For the atomic formula  $\varphi = R(\bar{x})$  let  $\varphi_G(\bar{x}, g) := R(\bar{x}, g)$ .
- If  $\varphi(\bar{x}) = (\psi \wedge \theta)(\bar{x})$  set  $\varphi_G(\bar{x}, g)$  as

$$\exists g_1 \exists g_2 \left( \psi_G(\bar{x}, g_1) \wedge \theta_G(\bar{x}, g_2) \wedge \bigwedge_{1 \leq i \neq j \leq 2} ((g_i \leq g_j) \rightarrow (g = g_i)) \right).$$

- If  $\varphi(\bar{x}) = (\psi_1 \rightarrow \psi_2)(\bar{x})$  set  $\varphi_G(\bar{x}, g)$  as

$$\exists g_1 \exists g_2 \left( \bigwedge_{i=1}^2 \psi_{i_G}(\bar{x}, g_i) \wedge ((g_1 \leq g_2) \rightarrow (g = \infty)) \wedge ((g_1 > g_2) \rightarrow (g = g_2)) \right).$$

- If  $\varphi(\bar{x}) = (\psi * \theta)(\bar{x})$  then

$$\varphi_G(\bar{x}, g) := \exists g_1 \exists g_2 (\psi_G(\bar{x}, g_1) \wedge \theta_G(\bar{x}, g_2) \wedge (g = g_1 * g_2)).$$

- For  $\varphi(\bar{x}) = (\psi^{-1})(\bar{x})$  let  $\varphi_G(\bar{x}, g) := \exists g_1 (\psi_G(\bar{x}, g_1) \wedge (g = g_1^{-1}))$ .
- If  $\varphi(\bar{x}) = \forall y \psi(y, \bar{x})$  set  $\varphi_G(\bar{x}, g)$  as

$$\forall a \forall g_1 (\psi_G(a, \bar{x}, g_1) \rightarrow g \leq g_1) \wedge \forall g_2 (g \leq g_2 \rightarrow \exists a \exists g_3 (\varphi_G(a, \bar{x}, g_3) \wedge g_3 \leq g_2)).$$

- If  $\varphi(\bar{x}) = \exists y \psi(y, \bar{x})$  let  $\varphi_G(\bar{x}, g)$  be as follows

$$\forall a \forall g_1 (\psi_G(a, \bar{x}, g_1) \rightarrow g_1 \leq g) \wedge \forall g_2 (g_2 \leq g \rightarrow \exists a \exists g_3 (\varphi_G(a, \bar{x}, g_3) \wedge g_2 \leq g_3)).$$

□

□

Now, using Lemma 4.18 and downward Löwenheim-Skolem theorem in classical first-order logic, the following theorem establishes.

**Theorem 4.19.** *(Downward Löwenheim-Skolem) Let  $\mathcal{M}$  be a  $\tau$ -structures and  $A \subseteq M$ . There is an elementary substructure  $\mathcal{N}$  of  $\mathcal{M}$  such that  $A \subseteq N$  and  $|N| \leq |A| + \|\tau\|$ .*

In classical model theory, an abstract elementary class (AEC) is a class of structures  $\mathfrak{A}$  with a partial ordering  $\prec_{\mathfrak{A}}$  which satisfies the following properties:

- (1) If  $\mathcal{M} \prec_{\mathfrak{A}} \mathcal{N}$  then  $M \subseteq N$ .
- (2)  $\mathfrak{A}$  is closed under isomorphism.
- (3) (Tarski-Vaught property) If  $\mathcal{M}_1 \prec_{\mathfrak{A}} \mathcal{N}$  and  $\mathcal{M}_2 \prec_{\mathfrak{A}} \mathcal{N}$  and  $M_1 \subseteq M_2$ , then  $\mathcal{M}_1 \prec_{\mathfrak{A}} \mathcal{M}_2$ .
- (4) (Union of chain axiom) If  $\mathcal{M}_1 \prec_{\mathfrak{A}} \mathcal{M}_2 \prec_{\mathfrak{A}} \dots$ , then  $\bigcup_{i=1}^{\infty} \mathcal{M}_i \in \mathfrak{A}$ . Furthermore, if  $\mathcal{M}_i \prec_{\mathfrak{A}} \mathcal{N}$ , for all  $i \geq 1$ , then  $\bigcup_{i=1}^{\infty} \mathcal{M}_i \prec_{\mathfrak{A}} \mathcal{N}$ .
- (5) The downward Löwenheim-Skolem property.

For example, the class of all models of a first order theory  $T$ , with the usual notion of "elementary substructure" forms an abstract elementary class. The above discussions together with the following two lemmas show that the class of exhaustive models of a theory  $T$  forms an abstract elementary class,

and the class of exhaustive ultrametric models of a theory  $T$  forms an abstract elementary class with amalgamation property.

**Lemma 4.20.** *Let  $(f, T) : \mathcal{M} \cong \mathcal{N}$ ,  $(g, S) : \mathcal{M}' \cong \mathcal{N}'$ ,  $\mathcal{M} \prec \mathcal{M}'$  and  $f \subseteq g$ . Then  $\mathcal{N} \prec \mathcal{N}'$ .*

*Proof.* Assume that  $U$  be an strict order preserving group homeomorphism such that  $U(\varphi^{\mathcal{M}}(\bar{a})) = \varphi^{\mathcal{M}'}(\bar{a})$  for any formula  $\varphi(\bar{x})$  and any  $\bar{a} \subseteq M$ . One could easily see that  $W = S \circ U \circ T^{-1}$  is an strict order preserving group homeomorphism such that  $W(\varphi^{\mathcal{N}}(\bar{b})) = \varphi^{\mathcal{N}'}(\bar{b})$  for any formula  $\varphi(\bar{x})$  and any  $\bar{b} \subseteq N$ .  $\square$

The Tarski-Vaught property is hold when the models are exhaustive.

**Lemma 4.21.** *Let  $\mathcal{M}_1 \prec \mathcal{N}$  and  $\mathcal{M}_2 \prec \mathcal{N}$ ,  $M_1 \subseteq M_2$ , and  $\mathcal{M}_i$  be exhaustive for  $i = 1, 2$ . Then  $\mathcal{M}_1 \prec \mathcal{M}_2$ .*

*Proof.* Straightforward.  $\square$

So we have the following result.

**Corollary 4.22.** *The class of exhaustive models of a theory  $T$  forms an abstract elementary class.*

Now, as an application of the amalgamation property and the elementary chain property, we prove the Robinson consistency theorem for AG $\forall$ .

**Theorem 4.23.** *Suppose  $\{T_i\}_{i=1,2}$  be  $m$ -satisfiable  $\tau_e^i$ -theories. Suppose also  $\tau_{\cap} = \tau_e^1 \cap \tau_e^2$  and  $\tau_{\cup} = \tau_e^1 \cup \tau_e^2$ . If  $T = T_1 \cap T_2$  is a linear complete  $\tau_{\cap}$ -theory, then  $T_1 \cup T_2$  is an  $m$ -satisfiable  $\tau_{\cup}$ -theory.*

*Proof.* Assume that  $\mathcal{M}_0$  and  $\mathcal{N}_0$  are two exhaustive ultrametric models of  $T_1$  and  $T_2$ , respectively. Since  $T$  is a linear complete  $\tau_{\cap}$ -theory, by an easy argument as in the proof of the Amalgamation Theorem we see that  $\Sigma_0 = \text{eddiag}(\mathcal{M}_0) \cup \text{eddiag}(\mathcal{N}_0)$  is an  $m$ -satisfiable  $\tau_{\cup}$ -theory. Let  $\mathcal{N}'_0$  be an exhaustive ultrametric model of  $\Sigma_0$ . So, one can find the elementary embeddings  $(f', T') : \mathcal{M}_0 \hookrightarrow_{\tau_{\cap}} \mathcal{N}'_0$  and  $(g, S) : \mathcal{N}_0 \hookrightarrow_{\tau_e^2} \mathcal{N}'_0$ .

Now, we claim that:

**Claim 1.** There is an exhaustive ultrametric elementary  $\tau_e^2$ -extension  $\mathcal{N}_1$  of  $\mathcal{N}_0$  and elementary embedding  $(f_0, T_0) : \mathcal{M}_0 \hookrightarrow_{\tau_{\cap}} \mathcal{N}_1$ .

**Proof of claim 1.** Since  $\mathcal{N}'_0 \models \text{eddiag}(\mathcal{N}_0)$ , for each  $b \in N_0$  let  $b' \in N'_0$  be the interpretation of  $c_b \in \tau_e^2(N_0)$ . Let  $N_1$  be the set obtained by replacing each  $b' \in N'_0$  by  $b \in N_0$ . Now,  $N_0 \subseteq N_1$ . Interpret all symbols of  $\tau_e^2$  in  $\mathcal{N}_1$  in the same manner as  $\mathcal{N}'_0$ . Then the function  $h$  defined by  $h(b) = b'$  for  $b \in N_0$  and  $h(x) = x$  for  $x \in N'_0 \setminus N_0$  together with the function  $S : G_{\mathcal{N}'_0} \rightarrow G_{\mathcal{N}'_0}$  is an isomorphism from  $\mathcal{N}_0$  onto  $\mathcal{N}_1$ . Furthermore, by setting  $f_0 = f' \circ h$  and  $T_0 = T' \circ S$  we have an elementary embedding  $(f, T)$  from  $\mathcal{M}_0$  into  $\mathcal{N}_1$ .  $\square_{\text{claim1}}$

Now we have  $\mathcal{M}_0 \models T_1$ ,  $\mathcal{N}_1 \models T_2$ , and an elementary embedding  $(f_0, T_0) : \mathcal{M}_0 \hookrightarrow_{\tau_{\cap}} \mathcal{N}_1$ . One could further strengthen the argument of the above claim to show that:

**Claim 2.** There is an exhaustive ultrametric elementary extension  $\mathcal{M}_1$  of  $\mathcal{M}_0$  and elementary embedding  $(g_1, S_1) : \mathcal{N}_1 \hookrightarrow_{\tau_{\cap}} \mathcal{M}_1$  such that  $(g_1 \circ f_0)(m) = m$  for each  $m \in M_0$ .

Applying claim 2, we get the following diagram of exhaustive ultrametric structures in which for each  $i \geq 0$ ,  $\mathcal{N}_{i+1}$  is an elementary  $\tau_e^2$ -extension of  $\mathcal{N}_i$ ,  $\mathcal{M}_{i+1}$  is an elementary  $\tau_e^1$ -extension of  $\mathcal{M}_i$ ,  $(f_i, T_i) : \mathcal{M}_i \hookrightarrow_{\tau_{\cap}} \mathcal{N}_{i+1}$  is an elementary embedding,  $(g_i, S_i) : \mathcal{N}_i \hookrightarrow_{\tau_{\cap}} \mathcal{M}_i$  is an elementary embedding,  $(g_{i+1} \circ f_i)(m) = m$  for each  $m \in M_i$ , and  $(f_{i+1} \circ g_{i+1})(n) = n$  for each  $n \in N_{i+1}$ .

$$\begin{array}{cccccccc}
 \mathcal{M}_0 & \prec & \mathcal{M}_1 & \prec & \dots & \prec & \mathcal{M}_i & \prec & \mathcal{M}_{i+1} & \prec & \dots \\
 & \searrow^{f_0} & \uparrow^{g_1} & \searrow^{f_1} & \dots & & \uparrow^{g_i} & \searrow^{f_i} & \uparrow^{g_{i+1}} & \searrow^{f_{i+1}} & \dots \\
 \mathcal{N}_0 & \prec & \mathcal{N}_1 & \prec & \dots & \prec & \mathcal{N}_i & \prec & \mathcal{N}_{i+1} & \prec & \dots
 \end{array}$$

Figure 1.

Now, let  $\mathcal{A}$  and  $\mathcal{B}$  be the reductions of  $\bigcup_{i=0}^{\infty} \mathcal{M}_i$  and  $\bigcup_{i=0}^{\infty} \mathcal{N}_i$  to the language  $\tau_{\cap}$ . Define  $f : A \rightarrow B$  by

$$f(a) = b \text{ if and only if for some } i \geq 0, f_i(a) = b$$

and  $T : G_{\mathcal{A}} \rightarrow G_{\mathcal{B}}$  by

$$T[\varphi(c_{m_1}, \dots, c_{m_k})] = [\varphi(c_{f(m_1)}, \dots, c_{f(m_k)})] \text{ for each } \tau_{\cap}\text{-formula } \varphi(x_1, \dots, x_k).$$

It is straightforward to see that  $f$  is a well-defined function and moreover  $(f, T) : \mathcal{A} \cong_{\tau_{\cap}} \mathcal{B}$  is an isomorphism.

Now, let  $\mathcal{D}$  be a  $\tau_{\cup}$ -structure whose underlying universe is  $A$  together with the following interpretations for the language symbols:

- for each  $s \in \tau_{\cap}$ ,  $s^{\mathcal{D}} = s^{\mathcal{A}}$ ,
- for each  $s \in \tau_e^1 \setminus \tau_{\cap}$ ,  $s^{\mathcal{D}} = s^{\bigcup_{i=0}^{\infty} \mathcal{M}_i}$ ,
- for each constant symbol  $s \in \tau_e^2 \setminus \tau_{\cap}$ ,  $s^{\mathcal{D}} = f(s^{\bigcup_{i=0}^{\infty} \mathcal{M}_i})$ ,
- for each function symbol  $s \in \tau_e^2 \setminus \tau_{\cap}$ ,  
 $(s(m_1, \dots, m_k))^{\mathcal{D}} = s^{\bigcup_{i=0}^{\infty} \mathcal{N}_i}(f(m_1), \dots, f(m_k))$ ,
- for each predicate symbol  $s \in \tau_e^2 \setminus \tau_{\cap}$ ,  
 $(s(m_1, \dots, m_k))^{\mathcal{D}} = T^{-1}\left(s^{\bigcup_{i=0}^{\infty} \mathcal{N}_i}(f(m_1), \dots, f(m_k))\right).$

Then,  $\mathcal{D} \models T_1 \cup T_2$  and the proof is complete.  $\square$

## 5. CONCLUDING REMARKS

We conclude the paper by proposing the following question.

**Question:** Is it true that any satisfiable sentence in additive Gödel logic, has a standard model?

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